Tutorial 5 : Selected problems of Assignment 6

Leon Li

24/10/2018

Recall the completion theorem:

Def Let (X,d) be a metric space. A completion of (X,d) is a metric space (Y,ρ) together with a map $\Phi: X \to Y$ such that

① (Y,ρ) is complete.

② Φ is an isometric embedding, i.e.

$$\forall x, x' \in X, \ d(x,x') = \rho(\Phi(x), \Phi(x'))$$

③ $\overline{\Phi(X)} = Y$

Thm Every metric space (X,d) has a completion.

Proof: Step 1: let $(C^b(X), \|\cdot\|_\infty)$ be the normed space of real-valued bounded continuous functions on X. Show that it is complete. (Q1)

Step 2: Construct an isometric embedding $\Phi: X \to C^b(X)$ (Q2)

Step 3: Define $Y := \overline{\Phi(X)} \subseteq C^b(X)$ with induced norm $\|\cdot\|_\infty$

then $(Y, \Phi: X \to Y)$ is a completion of (X,d).

Q1) (HW6, Q9) Show that $(C^b(X), \|\cdot\|_\infty)$ is complete.

Sol) Let $(f_n) \subseteq C^b(X)$ be a Cauchy sequence.

$\therefore \forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall m, n \geq N$, $\|f_n - f_m\|_\infty < \varepsilon$

$\therefore \forall x \in X$, $|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty < \varepsilon$

$\therefore \forall x \in X$, $(f_n(x)) \subseteq \mathbb{R}$ is a Cauchy sequence.

By completeness of $\mathbb{R}$, there exists a unique $y_x \in \mathbb{R}$ s.t. $\lim_n f_n(x) = y_x$

Define $f: X \to \mathbb{R}$ by $f(x) := y_x$.

We first show that f_n converges to f uniformly:

Using above notations, $|f_n(x) - f_m(x)| < \varepsilon$

Take $m \to +\infty$: $|f_n(x) - f(x)| \leq \varepsilon$, $\forall n \geq N, \forall x \in X$

$\therefore$ f_n converges to f uniformly on X.

Therefore, by Exchange Theorem, f is bounded continuous.

$\therefore f \in C^b(X)$. Also, as $\lim_n \|f_n - f\|_\infty = 0$, $\lim_n f_n = f$

$\therefore (f_n)$ converges in $C^b(X)$, hence $(C^b(X), \|\cdot\|_\infty)$ is complete.

Q2) (HW6, Q11) Construct an isometric embedding $\Phi : X \to C^b(X)$.

Sol) Fix $p \in X$; $\forall x \in X$, define $f_x : X \to \mathbb{R}$ by

$$f_x(z) := d(z,x) - d(z,p)$$

We first show that $f_x \in C^b(X)$:

(i) Bounded : $\forall z \in X$, $|f_x(z)| = |d(z,x) - d(z,p)| \leq d(x,p)$

(ii) Uniformly Continuous : $\forall z, z' \in X$,

$$|f_x(z) - f_x(z')| = |(d(z,x) - d(z,p)) - (d(z',x) - d(z',p))|$$

$$\leq |d(z,x) - d(z',x)| + |d(z',p) - d(z,p)|$$

$$\leq d(z,z') + d(z,z') = 2d(z,z')$$

$\therefore \forall \varepsilon > 0$, Choose $\delta = \frac{\varepsilon}{2} > 0$, then $\forall z, z' \in X$ with

$d(z,z') < \delta$, $|f_x(z) - f_x(z')| \leq 2d(z,z') < \varepsilon$.

$\therefore \forall x \in X$, $f_x \in C^b(X)$.

Define $\Phi : X \to C^b(X)$ by $\Phi(x) = f_x$.

It suffices to show that Φ is an isometric embedding:

$$\forall\, x, y \in X, \qquad \|f_x - f_y\|_\infty = d(x,y)$$

[$\leq$]: $\forall\, z \in X$, $|f_x(z) - f_y(z)| = |(d(z,x) - d(z,p)) - (d(z,y) - d(z,p))|$

$$= |d(z,x) - d(z,y)| \leq d(x,y).$$

$$\therefore \|f_x - f_y\|_\infty \leq d(x,y)$$

[$\geq$]: Take $z = y$, then $f_x(y) - f_y(y) = d(y,x) - d(x,x) = d(x,y)$

$$\therefore \|f_x - f_y\|_\infty \geq d(x,y)$$

Hence, $\|f_x - f_y\|_\infty = d(x,y)$

Q3) (HW6, Q10) Let $X = \mathbb{N}$ be the set of positive integers with metric $d : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ defined as

$$d(n, m) := \left| \frac{1}{n} - \frac{1}{m} \right|$$

a) Show that $(\mathbb{N}, d)$ is not complete.

b) Construct a completion $((Y, \rho), \Phi : \mathbb{N} \to Y)$

s.t. $Y \setminus \Phi(\mathbb{N})$ is a singleton set.

Sol: a) Consider $(x_n) = (n) \subseteq \mathbb{N}$.

① (n) is a Cauchy sequence: follows from the fact that $\left(\frac{1}{n}\right) \subseteq \mathbb{R}$ is a Cauchy sequence.

② (n) diverges in $(\mathbb{N}, d)$: Suppose it converges to $n_0 \in \mathbb{N}$, then choose $\varepsilon = \frac{1}{2n_0^2}$, $\exists\, N \in \mathbb{N}$ s.t. $\forall\, n \geq N$,

$$d(n, n_0) = \left| \frac{1}{n} - \frac{1}{n_0} \right| < \frac{1}{2n_0^2},$$ which is a contradiction

by choosing $n = \max\{N, n_0 + 1\}$, as

$$\left| \frac{1}{n} - \frac{1}{n_0} \right| \geq \frac{1}{n_0} - \frac{1}{n_0+1} = \frac{1}{n_0(n_0+1)} \geq \frac{1}{2n_0^2}$$

b) Define $Y := \mathbb{N} \cup \{\infty\}$ with metric $\rho : Y \times Y \to \mathbb{R}$

defined as (i) $\rho(n,m) = d(n,m) = \left|\frac{1}{n} - \frac{1}{m}\right|, \forall n,m \in \mathbb{N}$

(ii) $\rho(n,\infty) = \frac{1}{n} = \rho(\infty, n), \forall n \in \mathbb{N}$

(iii) $\rho(\infty,\infty) := 0$

then (Y,ρ) is a metric space.

(i) (Y,ρ) is complete: define an isometric embedding

$$\Psi : Y \longrightarrow \mathbb{R} \text{ by } \begin{cases} \Psi(n) = \frac{1}{n}, & n \in \mathbb{N} \\ \Psi(\infty) = 0 \end{cases}$$

then by definition of ρ, Ψ is an isometric embedding.

Hence, $\overline{\Psi(Y)} \subseteq \mathbb{R}$ is complete; Meanwhile

$\Psi : Y \xrightarrow[\cong]{} \Psi(Y) = \overline{\Psi(Y)}$ is an isometric isomorphism

$\therefore (Y,\rho)$ is complete.

(ii) Define $\Phi : \mathbb{N} \to Y$ by inclusion: $\Phi(n) = n$

then clearly Φ is an isometric embedding, and $\Phi(\mathbb{N}) = \mathbb{N} \subseteq Y$

$\therefore Y \setminus \Phi(\mathbb{N}) = \{\infty\}$ is a singleton set.